Outline of talk:

1) Introduction to higher dimensional dynamics in geometry.

2) Present a systemic approach to studying dynamics using the minimal model program

3) Give examples of the idea's in 2) being used in practice.

Over $\mathbb{C}$

Let X be a normal projective

Variety with "mild" singularities."

Mild means you can run some MMP.

Our ultimate goal: Let $f: X \dashrightarrow X$ be a

dominant rational map. We wish to

Study the behavior of $f, f \circ f, f \circ f \circ f, \ldots$ ect.

Write $f^n = \underbrace{f \circ f \circ \ldots \circ f}_{n\text{-times.}}$ $\qquad f, f^2, f^3, f^4, \ldots$

In order to study f and its iterates we would like some numerical invariants associated to $f, f^2, \ldots$ that measure its complexity.

ex: Consider $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by

$$(x,y) \mapsto (x^2y, y) \quad \text{on } \mathbb{A}^2.$$

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How can we measure the complexity of $f, f^2, \ldots$?

Basically can only look at $\deg(f)$ = max degree of the components = 3.

We want to study f and all iterates.

$f^2 = f((xy^2, y)) = ((xy^2)y^2, y) = (xy^4, y)$.

So $\deg f^2 = 5$. $\deg(f^2) \neq [\deg(f)]^2$

$f^3 = ((xy^4)y^2, y) = (xy^6, y)$, $\deg f^3 = 7$

Define $\lambda_1(f) = \lim_{n \to \infty} \deg(f^n)^{1/n}$ ← The first dynamical degree.

Limit exists

$$[2(n-1)+3]^{1/n} \to 1$$

In general $\deg(f^n) = 2(n-1)+3$ so

$\lambda_1(f) = 1.$

integer

More generally if A is a $k \times k$ matrix with rows $A_1, \dots, A_k$ we have

$f_A(x_1, \dots, x_k) = (x^{A_1}, \dots, x^{A_k})$, $f_A : \mathbb{P}^k \dashrightarrow \mathbb{P}^k$ and

Thm $\lambda_1(f) =$ spectral radius of A.

Our example was $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ eigenvalues 1

Studying the degree sequence of an arbitrary dominant rational map can be complicated.

Thm: (Bell, Diller, Jonsson, Krieger)

a) $\forall$ $K \geq 3$ there is a matrix $A \in SL_K(\mathbb{Z})$ and a birational involution $g_K : \underline{\mathbb{P}^K \dashrightarrow \mathbb{P}^K}$ s.t. $\lambda_1(\underline{g_K \circ f_A})$ is transcedental.

b) (Bell-Diller-Farve) There is a dominant rational map $\underline{f: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2}$ with $\lambda_1(f)$ transcedental.

To simplify the problem we will stick to <u>morphisms</u> $f: X \to X$.

ex: $X = \mathbb{P}^k$, $\underline{f: \mathbb{P}^k \to \mathbb{P}^k}$, then f is defined $k+1$ homogeneous polynomials of degree d that do not have a common vanishing locus. In this case $\underline{\lambda_1(f) = d}$.

Dynamical degrees:

Let X be a normal projective variety, and $f: X \to X$ a surjective morphism.

Choose an ample divisor H on X and define $\underline{\lambda_p(f)} = \underline{\lim_{n\to\infty} \left((f^n)^* H^p \cdot H^{\dim X - p} \right)^{1/n}}$ for $\underbrace{0 \le p \le \dim X.}$

The p^{th} dynamical degree.

So on $\mathbb{P}^k$ - $f^* H = dH$, when f is defined by degree d polynomials.

So $(f^n)^* H \cdot H^{k-1} = \underline{d^n}$ ⟵ giving $\lambda_1(f) = d$.

Dihn-Sibony.

When X is a compact-Kahler manifold and $f: X \dashrightarrow X$ a dominant rational map you can define $\lambda_p(f)$ in a similar way. We also have

$\lambda_p(f) = \lim_{n\to\infty} r_p(f^n)^{1/n}$ where $r_p(f^n)$ is the spectral radius of $f^n: H^{p,p}(X) \to H^{p,p}(X)$

When f is not a morphism $(f^n)^* \neq (f^*)^n$ on $H^{p,p}(X)$ but if f is a morphism $\lambda_p(f) =$ spectral radius of f^* acting on $H^{p,p}(X)$.

The sequence of dynamical degrees determines many dynamical properties of f, and its study goes back to Gromov in the 70's. Much of the more recent work has been done by

Dinh, Sibony, Truong, Zhang and many others. S_2

Problem (Already noted by Gromov)

Which projective var. actually admit surjective endomorphisms $f: X \to X$ with $\lambda_1(f) > 1$.

(not automorphisms)

1) $\mathbb{P}^n$

2) Abelian varieties

3) Toric varieties

Toric morphisms extending $(x_1, \dots, x_n) \mapsto (x_1^d, x_2^d, \dots, x_n^d)$

4) Products of such varieties

Classification for surfaces: (Nakayama)

X- smooth surface over $\mathbb{C}$. Let $f: X \to X$ be surjective of degree >1.

A) $K(X) = -\infty$ then X is toric or a ruled surface over a curve, C. If $g(C) \geq 2$ then the bundle splits after finite etale base change.

B) $K(X) = 0$, then f is unramified and X is an abelian surface or a hyper-elliptic surface.

C) $K(X) = 1$, X hyper elliptic with $\chi(\mathcal{O}_X) = 0$.

D) $K(X) = 2$ - none exist.

Upshot: Only in a few cases do we have explicit control over the behavior of morphisms.

A) $\mathbb{P}^n$

B) Abelian varieties

C) Toric Varieties?? ← I have some upcoming work here.

Here we can control the toric morphisms.

D) $\underbrace{\text{Polarized endomorphisms.}}$

We have canonical height functions for maps of $\mathbb{P}^n$.

Polarized endomorphisms:

This means $f^*H \equiv_{\text{lin}} dH$ for some integer $d>1$.

H-ample

Fahkruddin / Mumford

If $f: X \to X$ is a surjective morphism and $f^*H \equiv_{\text{lin}} dH$ with $d \geq 1$

then there is an embedding $i: X \hookrightarrow \mathbb{P}^N$ and $\phi: \mathbb{P}^N \to \mathbb{P}^N$ such that

f^n

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & \mathbb{P}^N \arrow[d, "\phi"] \\ X \arrow[r, hook, "i"'] & \mathbb{P}^N \end{tikzcd}$$

Reduction maps of projective space.

Moral: Except in some special cases we do not have an explicit description of surjective maps.

To get around this we can use the fact that $\underline{f^*: N^1(X) \to N^1(X)}$ preserves

f^* some linear automorphism.

1) The nef cone.

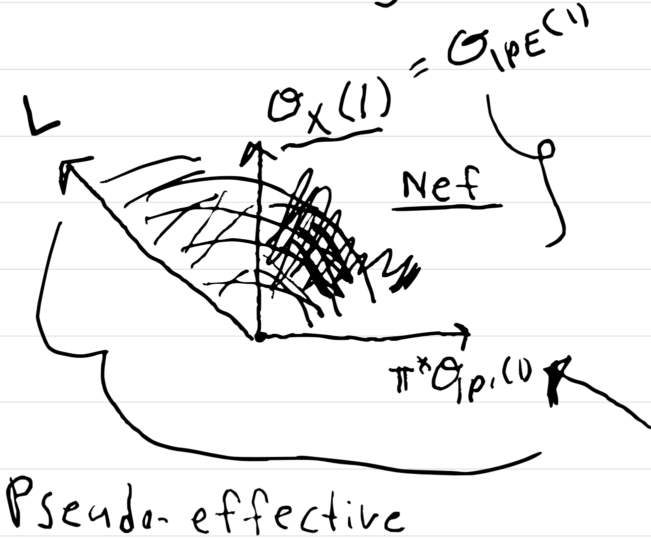
2) The pseudo-effective cone.

Ex: $X = \mathbb{P}(\overbrace{\mathcal{O} \oplus \mathcal{O}(r)}^{E}) \xrightarrow{\pi} \mathbb{P}^1$, $\quad r \geq 1$.

Suppose we have a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "g"'] & \mathbb{P}^1 \end{tikzcd}$$

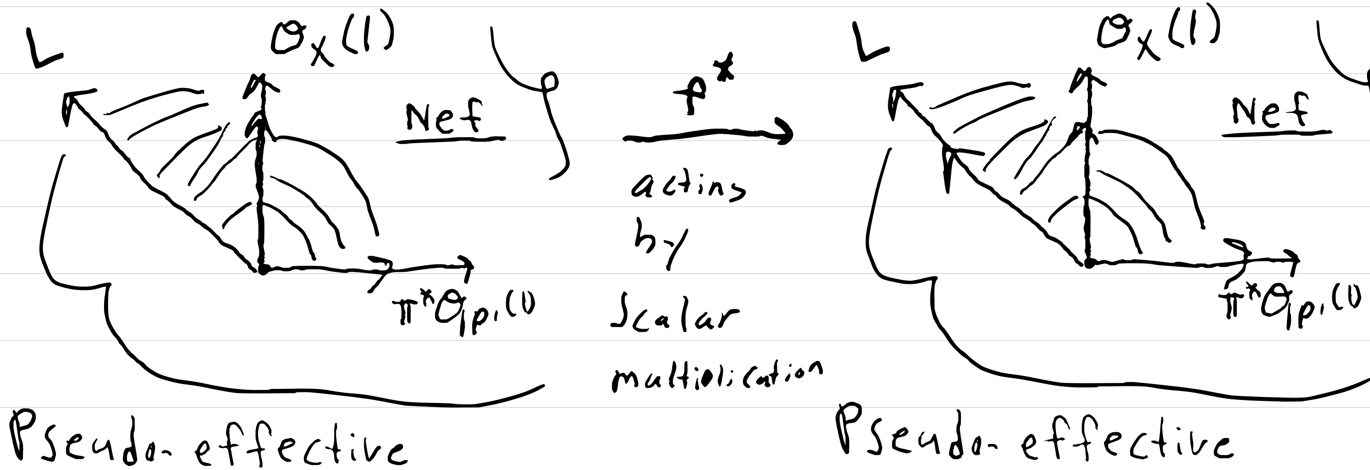
with f, g surjective.



$\underline{\mathcal{O}_X(1)}$ is nef and big but not ample.

$\mathcal{O}_X(1)^2 \neq 0$

$M \mapsto dM \leftarrow d = \deg g : \mathbb{P}^1 \to \mathbb{P}^1$



f^*M is ample $\iff$ M is ample
big $\iff$ M is big
nef $\iff$ M is nef
Pseudo-eff $\iff$ M is pseudo-eff

$\Rightarrow$ $\mathcal{O}_X(1)$, $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$, L all eigenvalues
$\Rightarrow$ $f^*M = dM \ \forall \ M \in \mathrm{Pic}(X)$.

Up-shot: All all eigenvalues have the same modulus.

Using some higher power results in Convex geometry geometry we can play Similar games in higher picard number.

Question: Let X be a smooth projective toric variety. Suppose that every extremal ray of $\mathrm{Nef}(X)$ lies on the boundary of $PE(X)$. Is $X = X_1 \times X_2$ where X_1, X_2 are smooth toric varieties?

The MMP: Finally

If we start with $f: X \to X$ a surjective morphism, perhaps we cannot understand f or the action of f^* on $\mathrm{Nef}(X)$ or $\mathrm{PE}(X)$ due to some complicated geometry of X, we would like to simplify $\mathrm{Nef}(X)$, $\mathrm{PE}(X)$.

Ideal situation: X - normal projective variety with terminal singularities, $f: X \to X$ and $\phi: X \to Y$ an extremal contraction.

1) If ϕ is divisorial, we obtain $g: Y \dashrightarrow Y$ a dominant rational map.

Try to Extend this to a morphism $g: Y \to Y$, after iterating f.

So we get

$$\begin{array}{ccc} X & \xrightarrow{f^n} & X \\ \phi \downarrow & & \downarrow \phi \\ Y & \xrightarrow{g} & Y \end{array}$$

Dream!!

$g^*: N^1(Y) \to N^1(Y)$ ← Hopefully easier.

2) If $\phi : X \to Y$ is small, then we have $\phi^+ : X \dashrightarrow X^+$ so we have

$$\begin{tikzcd} X \arrow[rr, dashed, "\phi^+"] \arrow[dr, "\phi"'] & & X^+ \arrow[dl, "\phi^+"] \\ & Y & \end{tikzcd}$$

$\underline{f^+ : X^+ \dashrightarrow X^+}$ a dominant rational map. Try to extend this to $\underline{f^+ : X^+ \to X^+}$ a morphism after iterating f. Dream 2)

3) If ϕ is a Mori-fiber space.

Thm: (Satriano/Lesieutre)
There is some $n \geq 1$ and $g: Y \to Y$
such that

$$\begin{tikzcd} X \arrow[r, "f^n"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes.

No need to try

For 1) and 2) we cannot always

Prove that we can extend f

to $Y \xrightarrow{g} Y$ or $f^+ : X^+ \to X^+$.

To help with this

Di-Qi Zhang introduced int-amplified endomorphisms.

Definition: Let X be a normal projective variety and let $f: X \to X$ be a surjective morphism. We say f is int-amplified if $f^*H - H$ is ample for some ample divisor H on X.

$\iff$ $f^*: N^1(X) \to N^1(X)$ has eigenvalues of modulus > 1.

examples: Any polarized morphism.

That is $f^* H = dH$, $d > 1$ H ample.

A) So any morphism $f: \mathbb{P}^n \to \mathbb{P}^n$ that is not an automorphism.

B) Multiplication by $[n]$ on an abelian variety. $n \geq 2$

C) Let X_Σ be toric. Let f be the equivariant morphism $X_\Sigma \to X_\Sigma$ induced by multiplication by n on $N \to N$, $\Sigma \subseteq N_{\mathbb{R}}$.

" $(x_1, .., x_n) \mapsto (x_1^d, .., x_n^d)$ "

Non-examples. Let C be an elliptic curve and F_r the unique rank r degree 0 vector bundle with a non-zero global section.

Thm: (N-Zotine 2023) $\mathbb{P}E \to C$ have no int-amplified endomorphisms.

Thm (Zhang-Meng)

If X is normal with terminal and X admits a single int-amplified morphism $I: X \to X$. If $f: X \to X$ is any other surjective morphism (it may not be int-amplified)

. If $\phi: X \to Y$ is an extremal contraction, then Dream1) and Dream2)

come true.

In other words, if X has <u>one</u> int-amplified morphism then we can use the MMP to study <u>all other</u> surjective morphisms.

Idea: $Sur(X)$ = monoid of surjective morphisms. Have $\text{Int-amp}(X) \subseteq Sur(X)$ sub-monoid of all int-amplified morphisms.

If $\text{int-amp}(X) \neq \emptyset$ then it imposes structure on all of $Sur(X)$.

Using this we have

Thm: (N) If X is $\mathbb{Q}$-factorial and rationally connected, and X admits $I: X \to X$ int-amplified.

Then <u>any</u> surjective endomorphism <u>$f: X \to X$</u> has a Zariski dense set of pre-periodic points.

(Fahkruddin proved this if f is int-amplified itself)

Start with X, terminal singularities and $I: X \to X$ int-amplified, rationally connected.

Start with $f: X \to X$ surjective map.

1) If X has a divisorial contraction we can find $n \geq 1$ s.t. we have a diagram

$$\begin{tikzcd} X \arrow[r, "f^n"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

1A) f has a dense set of pre-periodic points $\iff f^n$ does.

1B) f^n has a dense set of p.p. $\iff g$ does.

2) Similar story for flips.

$f^n : X \to X$ extends $\left\{ (f^+)^n : X^+ \to X^+ \right\}$

f^n has a dense set of pre-periodic points $\iff (f^+)^n$ does.

3)
$$\begin{tikzcd} X \arrow[r, "f^n"] \arrow[d, "\phi"'] & X \arrow[d, "\phi"] \\ Y \arrow[r, "g"'] & Y \end{tikzcd}$$

f^n sends fibers to fibers and we can check that on a general fiber f^n has many pre-periodic points.

$\Rightarrow$ conclude f^n has dense p.p. $\iff$ g does.